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**Density of Rational Points on K3 Surfaces over
Function Fields**

by

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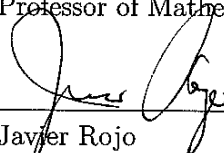
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ABSTRACT

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In this paper, we study sections of a Calabi-Yau threefold fibered over a curve by K3 surfaces. We show that there exist infinitely many isolated sections on certain K3 fibered Calabi-Yau threefolds and the subgroup of the Néron-Severi group generated by these sections is not finitely generated. This also gives examples of K3 surfaces over the function field F of a complex curve with Zariski dense F -rational points, whose geometric models are Calabi-Yau.

Furthermore, we also generalize our results to the cases of families of higher dimensional Calabi-Yau varieties with Calabi-Yau ambient spaces.

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To my parents

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Chapter 1

Introduction

1.1 Motivation from arithmetic geometry

Let K be a number field, i.e. a finite field extension of \mathbb{Q} . Given a collection of polynomial equations

$$f_1(x_1, \dots, x_n) = f_2(x_1, \dots, x_n) = \dots = f_m(x_1, \dots, x_n) = 0,$$

where $f_i(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$, we are interested in the solutions to these equations in K^n . In the modern geometric approach, one views these equations as describing an algebraic variety Y (or scheme) over K and the solutions in K^n as K -rational points on Y . Let us denote by $Y(K)$ the set of K -rational points on Y . There are some natural questions as following:

1. **Existence.** Is $Y(K)$ empty or not ?
2. **Zariski Density.** Is $Y(K)$ dense in the Zariski topology (i.e. the Zariski closure of $Y(K)$ is Y) ?

These question are very hard in general. However, in low-dimensional cases, much more is known.

Example 1.1.1. Let C be the Fermat curve over \mathbb{Q} given by the equation

$$x^n + y^n = 1.$$

It is well-known that C has a Zariski dense set of \mathbb{Q} -rational points when $n \leq 2$. If $n \geq 3$, there are only finitely many trivial solutions by Fermat's last theorem [?].

More generally, a uniform statement is expected depending only on the underlying geometry of Y . For instance, both questions have an affirmative answer for *Fano* varieties (i.e. the canonical class K_Y is negative) of dimension ≤ 2 after allowing a finite field extension K'/K . On the other hand, a conjecture of Lang and Vojta, confirmed in the case of curves and certain classes of higher dimensional varieties, predicts that Zariski density fails for varieties with positive canonical class (cf. [?]).

If the canonical class is trivial, the known examples are much less convincing. The simply connected case (e.g. Calabi-Yau varieties) remains mysterious, even in dimension two, i.e. the K3 surfaces.

1.2 K3 surfaces over function fields

Instead of the number field, we are interested in the Zariski density problem on K3 surfaces over the function field of a complex curve. We first recall some elements of the geometry of K3 surfaces.

1.2.1 K3 surfaces and geometric models

Let F be a field of characteristic 0.

Definition 1.2.1. A K3 surface S over F is a smooth projective geometrically integral surface such that the canonical class K_S is trivial and $H^1(S, \mathcal{O}_S) = 0$.

Example 1.2.1. Some explicit examples of K3 surfaces.

1. The double cover of \mathbb{P}_F^2 branched along a smooth sextic curve is a K3 surface of degree 2.
2. A smooth quartic hypersurface in \mathbb{P}_F^3 is a degree 4 K3 surface.

3. A smooth complete intersection of quadric and cubic hypersurfaces in \mathbb{P}_F^4 is a $K3$ surface of degree 6.

Now, we let $F = \mathbb{C}(B)$ be the function field of a complex curve B . Given a projective variety Y over F , one can construct a family

$$\pi : \mathcal{Y} \rightarrow B$$

over \mathbb{C} such that π is flat and proper and Y can be considered as the generic fiber of \mathcal{Y} . The projective family $\mathcal{Y} \rightarrow B$ is called a *geometric model* of Y . Each F -rational point on Y corresponds to a section $\text{Spec}(\mathbb{C}(B)) \rightarrow \mathcal{Y}$ of π , and thus to a rational map from B to \mathcal{Y} . Since \mathcal{Y} is proper, this extends uniquely to a section $\iota : B \rightarrow \mathcal{Y}$ of π .

Example 1.2.2. Let \mathbb{P}^1 be the projective line with coordinates (t_0, t_1) . The function field of $\mathbb{P}_{\mathbb{C}}^1$ is $\mathbb{C}(t)$, where $t = t_1/t_0$. The quadric surface $Y \subseteq \mathbb{P}_{\mathbb{C}(t)}^3$ determined by the equation

$$(1-t)x_0^2 + (2t-1)x_2^2 - 2x_1^2 + (2-2t^3)x_3^2 = 0$$

can be naturally considered as the generic fiber of the family of complex quadric surfaces

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3 \\ \downarrow \pi & \swarrow & \\ \mathbb{P}_{\mathbb{C}}^1 & & \end{array}$$

where \mathcal{Y} is the bidegree $(3, 2)$ hypersurface in $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3$ determined by the homogenous equation

$$(t_0^3 - t_1 t_0^2)x_0^2 + (2t_1 t_0^2 - t_0^3)x_2^2 - 2t_0^3 x_1^2 + (2t_0^3 - 2t_1^3)x_3^2 = 0.$$

The $\mathbb{C}(t)$ -rational point $(t, 1, t, 1)$ on Y can be viewed as a section $\iota : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathcal{Y}$ given by $\iota(t_0, t_1) = (t_0, t_1; t_1, t_0, t_1, t_0)$.

Let S be a $K3$ surface over the function field $\mathbb{C}(B)$ and $\mathcal{S} \rightarrow B$ its geometric model. Denote by $S(\mathbb{C}(B))$ the $\mathbb{C}(B)$ -rational points on S . The points in $S(\mathbb{C}(B))$ correspond to sections on $\mathcal{S} \rightarrow B$, which can be considered as certain curves. Our questions from arithmetic in §1.1 can be restated in classical geometry over \mathbb{C} as follows:

Question 1. (i). **Existence.** *Does there exist a section on $\mathcal{S} \rightarrow B$?*

(ii). **Zariski Density.** *Is the union of all sections on $\mathcal{S} \rightarrow B$ dense in the Zariski topology of \mathcal{S} ?*

1.2.2 Families of quartic surfaces

Let $\pi : \mathcal{S} \rightarrow B$ be a projective family of degree d $K3$ surfaces in $\mathbb{P}_{\mathbb{C}}^n$ over a smooth curve B and $\mathcal{O}_{\mathcal{S}}(1)$ the polarization of \mathcal{S} . Let $\mathfrak{Hilb}_{P_d(t)}(\mathbb{P}_{\mathbb{C}}^n)$ be the Hilbert scheme parametrizing all degree d surfaces in $\mathbb{P}_{\mathbb{C}}^n$ with Hilbert polynomial $P_d(t) = 2 + dt^2/2$. Then a local trivialization of $\pi_*\mathcal{O}_{\mathcal{S}}(1)$ induces a rational map $B \dashrightarrow \mathfrak{Hilb}_{P_d(t)}(\mathbb{P}_{\mathbb{C}}^n)$, which extends to a morphism

$$B \longrightarrow \mathfrak{Hilb}_{P_d(t)}(\mathbb{P}_{\mathbb{C}}^n). \quad (1.1)$$

The morphism (??) corresponds to a curve on $\mathfrak{Hilb}_{P_d(t)}(\mathbb{P}_{\mathbb{C}}^n)$.

For instance, the quartic surfaces in $\mathbb{P}_{\mathbb{C}}^3$ are determined by homogenous quartic polynomials in $\mathbb{C}[x_0, \dots, x_3]$, which are parameterized by $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^3}(4))) = \mathbb{P}_{\mathbb{C}}^{34}$. The simplest curve in $\mathbb{P}_{\mathbb{C}}^{34}$ is a line, which can correspond to a *pencil* of quartic surfaces. (cf. [?]).

Example 1.2.3. A pencil \mathcal{S} of quartic surfaces can be written as

$$\begin{array}{c} \mathcal{S} = \{sf + tg = 0\} \subset \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3 \\ \downarrow \pi \\ \mathbb{P}_{\mathbb{C}}^1 \end{array}$$

for some quartic polynomials $f, g \in \mathbb{C}[x_0, x_1, x_2, x_3]$. The base locus

$$C = \{(x_0, \dots, x_3) \mid f(x_0, \dots, x_3) = g(x_0, \dots, x_3) = 0\} \subset \mathbb{P}_{\mathbb{C}}^3$$

gives a one-parameter family of constant sections $\iota_c(s, t) = (s, t; c_0, \dots, c_3)$ for each point $c = (c_0, \dots, c_3) \in C$.

Definition 1.2.2. We say that a projective variety X is “very general” if it is chosen outside a union of countably many closed proper subsets of its parametrization space.

In [?], Hassett and Tschinkel have proved the following result:

Theorem 1.2.3. *Let $\mathcal{S} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ be a very general pencil of quartic surfaces. Then there exist infinitely many one-parameter families of sections on \mathcal{S} , whose union is Zariski dense in \mathcal{S} .*

We would like to refer the readers to [?] for more general results and details.

Conics in $\mathbb{P}_{\mathbb{C}}^{34}$. In chapter 2, we will investigate Question1 for families of degree four K3 surfaces over conics in \mathbb{P}^{34} . The geometric models involved are bidegree $(2, 4)$ hypersurfaces in $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3$, whose ambient spaces are Calabi-Yau threefolds.

Let X be a smooth hypersurface in $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3$ of bidegree $(2, 4)$. According to the geometry of Calabi-Yau threefolds, the sections in X are expected to be *isolated*, i.e. the space of embedded deformations in X is reduced and zero dimensional. For

instance, the bidegree $(2, 4)$ hypersurface X can be written as

$$X = \{t_0^2 f + t_0 t_1 g + t_1^2 h = 0\} \subseteq \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3$$

for some quartic polynomials $f, g, h \in \mathbb{C}[x_0, \dots, x_3]$. It only has finitely many sections of degree 0, which are isolated and given by the points in $\{f = g = h = 0\}$.

As in [?], we construct sections on $X \subseteq \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3$ by specializing X to a nodal hypersurface X_0 with infinitely many isolated sections lying in the smooth locus of X_0 . Thus we can get infinitely many sections on X by using deformation theory.

To prove the Zariski density, we need to show that there does not exist any surface in X containing these sections. It is not obvious in this case since there are only countably many sections on X . We will propose an approach to this problem using Hodge theory in the next section.

Remark 1.2.4. *Given a bidegree $(2, 4)$ hypersurface $X \hookrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3$, we expect only finitely many sections of degree n on this Calabi-Yau threefold for each integer $n \geq 0$. For example, there are only 64 sections of degree 0 for general X . An interesting question is to count the number of sections for each degree.*

Let $\beta = (d_1, d_2)$ be a homology class in $H_2(X, \mathbb{Z})$. The genus zero Gromov-Witten invariants $N_{0, \beta}^X$ on X are determined by a hypergeometric series, which is understood rigorously (cf. [?]). By a direct computation, we get

$$N_{0, (1, d)}^X = 64, 6912, 2178680, \dots$$

for $d = 0, 1, 2, \dots$. It is natural to ask whether these numbers are enumerative.

High degree rational curves in $\mathbb{P}_{\mathbb{C}}^{34}$. One can certainly consider the families of quartic surfaces over high degree rational curves in $\mathbb{P}_{\mathbb{C}}^{34}$. More explicitly, let X be a very general hypersurface in $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3$ of bidegree $(d, 4)$, which corresponds to a degree

d rational curve C_d in $\mathbb{P}_{\mathbb{C}}^{34}$. It is natural to ask whether Question 1 have positive answers.

Unfortunately, it seems that conics ($d=2$) is the highest degree case where Question 1(ii) has a positive answer. Even the existence of the sections on $X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is expected to fail when $d \geq 3$. To see why, we give a heuristic approach as follows:

One can specialize X as a union of pencils $\tilde{X} = \bigcup_{i=1}^d X_i$ meeting transversely along fibers, which corresponds to a chain of lines $\tilde{C}_d = L_1 \cup L_2 \dots \cup L_d$ such that $L_i \cap L_{i+1}$ is a single point q_i for $i = 1, \dots, d-1$. Then it suffices to show that there is no *stable section* on $\pi : \tilde{X} \rightarrow \tilde{C}_d$ for \tilde{C}_d very general. Here the definition of *stable sections* is given in [?]. Roughly speaking, a stable section on $\tilde{X} \rightarrow \tilde{C}_d$ can be interpreted as a stable curve of genus zero on \tilde{X} and meets the generic fiber of $\tilde{X} \rightarrow \tilde{C}_d$ at a single point.

It is expected that there are only countably many *stable sections* on $X_1 \cup X_2 \rightarrow L_1 \cup L_2$ for a very general chain $L_1 \cup L_2$, which corresponds to a countable set $\Sigma \subset \pi^{-1}(q_2)$. By [?], there exists only countably many one parameter families of sections on $X_3 \rightarrow L_3$ corresponding to countably many curves C_i in $\pi^{-1}(q_2)$. We can expect for a very general choice of L_3 , neither C_i passes through any point in Σ nor there exist rational curves on $\pi^{-1}(q_2)$ connecting Σ and C_i . We refer the reader to [?] §7 for similar discussions.

More generally, let $\mathfrak{X} \rightarrow \mathbb{P}_{\mathbb{C}}^{34}$ be the universal family of the parameter space of all quartic surfaces in $\mathbb{P}_{\mathbb{C}}^3$. In light of the results in [?] §5, we know that for a very general curve B in $\mathbb{P}_{\mathbb{C}}^{34}$ of genus g and degree d , if g is sufficiently large and $d \gg g$, then the one parameter family $\mathfrak{X} \times_{\mathbb{P}_{\mathbb{C}}^{34}} B \rightarrow B$ does not admit a section. Therefore, the best result we could hope for in general is:

Conjecture 1. *Let $X \rightarrow B$ be a family of quartic surfaces over a smooth projective*

curve B . Then there exist a smooth curve $C \rightarrow B$, such that the family $X \times_B C \rightarrow C$ has a Zariski dense set of sections.

1.3 Review of algebraic cycles and Hodge theory

In this section, we will briefly discuss the connection between Hodge theory and groups of various cycles classes.

1.3.1 Hodge structures and intermediate Jacobians.

Definition 1.3.1. An integral Hodge structure of weight m consists of a free abelian group $V_{\mathbb{Z}}$ of finite type and a decomposition

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q},$$

satisfying the Hodge symmetry condition $V^{p,q} = \overline{V^{q,p}}$.

The Hodge filtration $F^{\bullet}V_{\mathbb{C}}$ associated to such a Hodge structure is the decreasing filtration defined by

$$F^p V_{\mathbb{C}} = \bigoplus_{i \geq p} V^{i, m-i}.$$

If X is a smooth complex projective variety of dimension n , the m -th Betti cohomology group $H^m(X, \mathbb{Z})$ of X with integral coefficients carries an integral Hodge structure of weight m . The corresponding Hodge decomposition is given by

$$H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X),$$

where $H^{p,q}(X)$ can be identified as the q -th sheaf cohomology group of the sheaf of differential p -forms on X .

If $m = 2k - 1$ is odd, we define the k -th intermediate Jacobian

$$J^k(X) := \frac{H^{2k-1}(X, \mathbb{C})}{F^m H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbb{Z})}$$

as a compact complex torus. Using Poincaré duality, we have the identification

$$J^k(X) = \frac{(F^{n-k+1}H^{2n-2k+1}(X))^*}{H_{2n-2k+1}(X, \mathbb{Z})}.$$

When $k = 1$, $J^1(X)$ is nothing but the Picard variety of X . If X is a smooth projective threefold, we denote by $J(X)$ the 2-nd intermediate Jacobian of X .

1.3.2 Groups of algebraic cycles on threefolds.

Let X be a smooth projective variety of dimension n . An algebraic cycle of codimension k on X is a formal linear combination $Z = \sum n_i Z_i$ of irreducible reduced closed subvarieties Z_i of codimension k for some integer $n_i \in \mathbb{Z}$.

Definition 1.3.2. Let $\mathcal{Z}^i(X)$ be the group of algebraic cycles on X of codimension k . A cycle $Z \in \mathcal{Z}^k(X)$ is called algebraic equivalent to zero if there exists a finite collection $\{Y_i, f_i\}$, with Y_i irreducible subvarieties of X of codimension $(k-1)$ and f_i rational functions on Y_i , such that $Z = \sum_i \text{div}(f_i)$.

Remark 1.3.3. *There is an equivalent definition of rational equivalence. We say $Z \sim_{rat} Z'$ if there is a cycle V on $X \times \mathbb{P}^1$ flat over \mathbb{P}^1 , such that $V \cap X \times \{0\} = Z$ and $V \cap X \times \{\infty\} = Z'$.*

We define the k -th Chow group $CH^k(X)$ to be the quotient $\mathcal{Z}^k(X)/\sim_{rat}$. The simplest way to go from Chow groups to Hodge structures is to use the cycle class map:

$$\text{cl}_k : CH^k(X) \rightarrow H_{2n-2k}(X, \mathbb{Z}),$$

defined by

$$\sum n_i Z_i \rightarrow \sum n_i [Z_i]$$

where $[Z_i]$ is the fundamental class of Z_i , i.e. the Poincaré dual of the current integral on Z_i . Let $CH^k(X)_{hom}$ be the kernel of the cycle class map cl_k .

Next, let us introduce another group of algebraic cycle classes.

Definition 1.3.3. An algebraic cycle Z is said to be algebraic equivalent to Z' if there is a non-singular projective curve C , an element $\mathcal{Z} \in CH^k(X \times C)$ and points $x_1, x_2 \in C$ such that $i_1^* \mathcal{Z} - i_2^* \mathcal{Z} = Z - Z'$, where $i_j : X \hookrightarrow X \times C$ is given by $i_j(x) = (x, x_j)$.

It is easy to see that the rational equivalence relation is stronger than the algebraic equivalence relation. We denote by $\mathcal{A}^k(X)$ the group of codimension k algebraic cycles on X modulo algebraic equivalence and define the Griffiths group $\text{Griff}^k(X)_{hom}$ to be the quotient of $CH^k(X)_{hom}$ modulo algebraic equivalence.

Remark 1.3.4. If $k = 1$, the group $\mathcal{A}^1(X)$ is isomorphic to the Néron-Severi group $NS(X)$, which is a finitely generated abelian group.

In [?], Griffiths has introduced an Abel-Jacobi map

$$AJ_X^k : CH^k(X)_{hom} \rightarrow J^k(X),$$

sending each cycle Z with $[Z] = \partial W$ to

$$\int_W \in (F^{n-k+1} H^{2n-2k+1}(X))^* / H_{2n-2k+1}(X, \mathbb{Z}),$$

where W is a real chain of dimension $2n-2k+1$ well defined up to a $(2n-2k+1)$ -cycle.

When X is a general non-rigid Calabi-Yau threefold, C.Voisin [?] has shown that the Abel-Jacobi map AJ_X factors through the Griffiths group $\text{Griff}^2(X)$ and the Abel-Jacobi image of $\text{Griff}^2(X)$ is infinitely generated. As a result, $\mathcal{A}^2(X)$ is not finitely generated. Now let X be a bidegree $(2, 4)$ hypersurfaces in $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3$ and $\mathcal{A} \subseteq \mathcal{A}^2(X)$ be the subgroup generated by all sections on X . There is a natural question:

Question 2. *Is \mathcal{A} infinitely generated?*

A key fact is that Zariski density holds on X if Question 2 has a positive answer. We refer the reader to §2.6 for more details. In this paper, we will follow the method of Clemens [?] to prove the infinite generation of \mathcal{A} .

Chapter 2

Sections of Calabi-Yau threefolds with K3 fibration

2.1 Statement of main results

Let \mathbb{P}^n be the complex projective space of dimension n . Our first theorem is as following:

Theorem 1. *Let X be a very general hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ of bidegree $(2, 4)$. There exist countably many isolated sections ℓ_n on X with respect to the projection to \mathbb{P}^1 . Let \mathcal{A} be the subgroup of $\mathcal{A}^2(X)$ generated by the ℓ_i . Then \mathcal{A} is not finitely generated.*

As we discussed in §1.2, the density result follows from the infinite generation of \mathcal{A} .

Theorem 2. *The union of the sections on $X \rightarrow \mathbb{P}^1$ is Zariski dense in X .*

The rest of this chapter is organized as follows: In Section 2, we recall the Néron model theory on degenerations of intermediate Jacobians. In particular, we state a theorem describing the Néron models coming from geometry. Section 3 is devoted to showing the existence of infinitely many isolated sections on X . We will describe the construction of these section using specialization. In Section 4, we find a useful degeneration of our Calabi-Yau threefolds and study the the deformation theory of curves on the singular fiber of the degeneration. As an application of the result in §2, we compute the group of components of the Néron model associated to the

degeneration. The main theorems are proved in Section 5 and 6. In the last section, we extend our results to higher dimensional cases.

2.2 Preliminaries on Néron models

In this section, we briefly review some results [?] of Néron model theory for families of intermediate Jacobians coming from a variation of Hodge structure (VHS), which will be used later in this paper. For simplicity, our VHS arises from geometry and is parametrized by a complex disc.

2.2.1 Geometric setting.

Let X be a smooth projective variety of dimension $2k-1$. The intermediate Jacobian $J(X)$ of X is a compact torus defined as

$$J(X) = H^{2k-1}(X, \mathbb{C}) / (F^k H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbb{Z}))$$

where $F^k H^{2k-1}(X)$ is the Hodge filtration of $H^{2k-1}(X)$.

More generally, let Δ be a complex disc and let $\pi : \mathcal{X} \rightarrow \Delta$ be a *semistable degeneration*, that is:

- (1) \mathcal{X} is smooth of dimension $2k$;
- (2) π is projective, with the restriction $\pi : \mathcal{X}^* = \mathcal{X} \setminus \pi^{-1}(0) \rightarrow \Delta^*$ smooth, where $\Delta^* = \Delta \setminus \{0\}$;
- (3) the fiber $\mathcal{X}_0 = \pi^{-1}(0)$ is reduced with non-singular components crossing normally;
write $\mathcal{X}_0 = \cup X_i$.

Consider the VHS associated to the $(2k-1)$ th cohomology along the fibres of $\pi : \mathcal{X} \rightarrow \Delta^*$; then there is family of intermediate Jacobians

$$\mathcal{J} \rightarrow \Delta^*, \quad (2.1)$$

which forms an analytic fiber space with fiber $\mathcal{J}_s = J(\mathcal{X}_s), s \in \Delta^*$.

Because of the semistability assumption, the Monodromy theorem [?] implies that the monodromy transformation

$$T : H^{2k-1}(\mathcal{X}_s, \mathbb{Z}) \rightarrow H^{2k-1}(\mathcal{X}_s, \mathbb{Z})$$

is unipotent. In this situation, Green, Griffiths and Kerr [?] have constructed a slit analytic space $\bar{\mathcal{J}}(\mathcal{X}) \rightarrow \Delta$ such that

- the restriction $\bar{\mathcal{J}}(\mathcal{X})|_{\Delta^*}$ is $\mathcal{J} \rightarrow \Delta^*$;
- every *admissible normal function* (ANF) extends to a holomorphic section of $\bar{\mathcal{J}}(\mathcal{X}) \rightarrow \Delta$; here an ANF is a holomorphic section of (??) satisfying the admissibility condition (cf. [?] or [?] II.B).
- the fiber $\bar{\mathcal{J}}_0(\mathcal{X})$ inserted over the origin fits into an exact sequence

$$0 \rightarrow \mathcal{J}_0 \rightarrow \bar{\mathcal{J}}_0(\mathcal{X}) \rightarrow G \rightarrow 0, \quad (2.2)$$

where G is a finite abelian group and \mathcal{J}_0 is a connected, complex Lie group, considered as the identity component of $\bar{\mathcal{J}}_0(\mathcal{X})$.

The total space $\bar{\mathcal{J}}(\mathcal{X})$ is called the Néron model associated to \mathcal{X} .

Remark 2.2.2. *In fact, every ANF without singularities [?] extends to the identity component (cf. [?] II. A).*

Remark 2.2.3. *Kato, Nakayama and Usui have an alternate approach constructing Néron models via a log mixed Hodge theory[?], which is homeomorphic to the construction in [?]. (cf. [?])*

2.2.4 Abel-Jacobi map

Let $CH^k(X)_{hom}$ be the subgroup of the Chow group of X consisting of codimension k algebraic cycles which are homologically equivalent to zero. There is an Abel-Jacobi map

$$AJ_X : CH^k(X)_{hom} \rightarrow J(X) \quad (2.3)$$

defined in §1.3.2.

Returning to the semistable degeneration $\mathcal{X} \rightarrow \Delta$, given a codimension k algebraic cycle $\mathcal{Z} \subset \mathcal{X}$ with $Z_s = \mathcal{Z} \cdot \mathcal{X}_s \in CH^k(\mathcal{X}_s)_{hom}$ for $s \neq 0$, there is an associated admissible normal functions $\nu_{\mathcal{Z}}$ via the Abel-Jacobi map

$$\nu_{\mathcal{Z}}(s) = AJ_{\mathcal{X}_s}(Z_s), \quad s \in \Delta^*. \quad (2.4)$$

(cf. [?] III)

Furthermore, the associated function $\nu_{\mathcal{Z}}$ will extend to the identity component of $\bar{\mathcal{J}}(\mathcal{X})$ if \mathcal{Z} is cohomological to zero in \mathcal{X} .

2.2.5 Threefold case.

With the notation above, now we assume that $\mathcal{X} \rightarrow \Delta$ is a semistable degeneration of projective threefolds, and denote by $\bar{\mathcal{X}}$ a smooth projective variety containing \mathcal{X} as an analytic open subset.

In this situation, we have a precise description of the group of components G via an intersection computation.

Theorem 2.2.6. ([?] Thm.III. C.6) For any multi-index $I = (i_0, \dots, i_m)$, $|I| = m + 1$, let

$$\begin{aligned} Y_I &= \bigcap_{i \in I} X_i \\ Y^{[m]} &= \bigsqcup_{|I|=m+1} Y_I. \end{aligned} \tag{2.5}$$

Assuming that all the cohomology groups of $Y^{[m]}$ are torsion free, then the natural map $j : Y^{[0]} \rightarrow \bar{\mathcal{X}}$ induces a sequence of maps

$$H_4(Y^{[0]}, M) \xrightarrow{j_*^M} H_4(\bar{\mathcal{X}}, M) \cong H^4(\bar{\mathcal{X}}, M) \xrightarrow{j_M^*} H^4(Y^{[0]}, M) \cong H_2(Y^{[0]}, M)$$

where $M = \mathbb{Z}$ or \mathbb{Q} , and the composition gives the morphism

$$\mu_M : \bigoplus_{i=1}^m H_4(X_i, M) \rightarrow \bigoplus_{i=1}^m H_2(X_i, M).$$

Then there is an identification of the group G in (??),

$$G = \frac{(\text{Im } \mu_{\mathbb{Q}})_{\mathbb{Z}}}{\text{Im } \mu_{\mathbb{Z}}}. \tag{2.6}$$

Furthermore, the extension of the admissible normal function $\nu_{\mathcal{Z}}$ (??) maps to the component corresponding to the class $[Z_0]$ in $\bigoplus_{i=1}^m H_2(X_i, \mathbb{Z})$.

Remark 2.2.7. A similar result holds for a degeneration of curves. But when $\dim \mathcal{X}_s > 3$, the identification (??) may fail (cf. [?]).

2.3 Construction of sections on K3-fibered Calabi-Yau three-folds

In this section, our aim is to show the existence of isolated sections on a general bidegree $(2, 4)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ with respect to the projection to \mathbb{P}^1 . We

begin with the construction of a hypersurface X_0 with at worst nodes as singularities containing infinitely many isolated sections.

Lemma 2.3.1. *There exists a hypersurface $X_0 \subset \mathbb{P}^1 \times \mathbb{P}^3$ with finitely many nodes, such that X_0 admits an infinite collection of sections $\{\ell_n\}$ with respect to the projection $X_0 \rightarrow \mathbb{P}^1$. Moreover, each ℓ_n lies in the smooth locus of X_0 and is infinitesimally rigid.*

Proof. Let $S \rightarrow \mathbb{P}^1$ be a smooth rational elliptic surface, obtained by blowing up \mathbb{P}^2 along nine base points of a general pencil of cubic curves. This was first studied by Nagata [?], who showed there are infinitely many exceptional curves of the first kind, and each of them yields a section ℓ_n of $S \rightarrow \mathbb{P}^1$.

We have a natural embedding $S \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ and choose a smooth surface $H \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(1, 1)$ meeting S transversally in $\mathbb{P}^1 \times \mathbb{P}^2$.

Let $x = (t_0, t_1; x_0, \dots, x_3)$ be the coordinates of $\mathbb{P}^1 \times \mathbb{P}^3$. Consider $\mathbb{P}^1 \times \mathbb{P}^2$ as a hyperplane of $\mathbb{P}^1 \times \mathbb{P}^3$ defined by $x_3 = 0$. Let $|\mathfrak{L}|$ be the linear system of bidegree $(2, 4)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$ containing S and H . Then a general member in $|\mathfrak{L}|$ will be a singular hypersurface with finitely many nodes contained in $S \cap H$.

More explicitly, assume that S is defined by $q(x) = x_3 = 0$, while H is given by the equations $l(x) = x_3 = 0$ for some polynomial $q(x)$ of bidegree $(1, 3)$ and $l(x)$ of bidegree $(1, 1)$.

Then a hypersurface $X_0 \in |\mathfrak{L}|$ is given by an equation

$$l(x)q(x) + x_3f(x) = 0 \tag{2.7}$$

for some bidegree $(2, 3)$ polynomial $f(x)$. The singularities of X_0 are eighteen nodes defined by

$$l(x) = f(x) = x_3 = q(x) = 0. \tag{2.8}$$

for a generic choice of $f(x)$ by Bertini's theorem.

For each n , the space of X_0 containing a node on ℓ_n is only a finite union of hypersurfaces in $|\mathfrak{L}|$. Then we can ensure that no node of X_0 lies on $\{\ell_n\}$ for a generic choice of $f(x)$ avoiding countably many hypesurfaces in $|\mathfrak{L}|$.

Furthermore, since $\ell_n^2 = -1$ in S , then $\mathcal{N}_{\ell_n/S} = \mathcal{O}_{\ell_n}(-1)$. Then the normal bundle exact sequence

$$0 \rightarrow \mathcal{O}_{\ell_n}(-1) \rightarrow \mathcal{N}_{\ell_n/X_0} \rightarrow \mathcal{O}_{\ell_n}(-1) \rightarrow 0 \quad (2.9)$$

implies that

$$\mathcal{N}_{\ell_n/X_0} \cong \mathcal{O}_{\ell_n}(-1) \oplus \mathcal{O}_{\ell_n}(-1).$$

This proves the infinitesimal rigidity.

The following result follows from the above lemma and deformation theory.

Theorem 2.3.2. *For a general bidegree $(2, 4)$ hypersurface X in $\mathbb{P}^1 \times \mathbb{P}^3$, there exist infinitely many sections $\{\ell_n\}$ on X with respect to the projection $X \rightarrow \mathbb{P}^1$ such that ℓ_n is infinitesimally rigid in X .*

Proof. From the above lemma, the rational curves ℓ_n in X_0 are stable under deformations by the Kodaira stability theorem [?]. This implies that the relative Hilbert scheme parameterizing the pair (ℓ, X) , $\ell \subset X$ is smooth over the deformation space of X at (ℓ_n, X_0) , and hence dominating. These sections ℓ_n deform to nearby neighborhoods of X_0 . Although X_0 is singular, we can restrict everything to the smooth locus of X_0 to ensure the argument still applies.

Furthermore, the fibration $\pi : X_0 \rightarrow \mathbb{P}^1$ is given by the linear system $|\pi^* \mathcal{O}_{\mathbb{P}^1}(1)|$. Since $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ has no higher cohomology, it deforms with X_0 and the dimension of $|\pi^* \mathcal{O}_{\mathbb{P}^1}(1)|$ is constant by semicontinuity. Thus the fibration will be preserved under deformation. Note that the deformation of ℓ_n meets the generic fiber of π at one point;

it follows that the deformation of ℓ_n remains to be a section in a general deformation of X_0 .

Throughout this paper, by abuse of the notation, we continue to denote $\ell_n \subset X$ by the section obtained from the deformation of $\ell_n \subset X_0$.

Remark 2.3.3. *Our method constructs infinitely many isolated rational curves of bidegree $(1, d)$ on a K3-fibered Calabi-Yau threefold in $\mathbb{P}^1 \times \mathbb{P}^N$. For examples, the exceptional divisors of S give degree 0 sections on $X_0 \rightarrow \mathbb{P}^1$, which are of type $(1, 0)$. See [?] for the existence of isolated rational curves of bidegree $(0, d)$ on K3-fibered Calabi-Yau threefolds in $\mathbb{P}^1 \times \mathbb{P}^N$ for every integer $d \geq 1$.*

2.4 The degeneration of Calabi-Yau threefolds

In this section, we will study the degeneration of our Calabi-Yau threefolds and the deformation theory of sections on the degenerations.

2.4.1 An important degeneration.

Lemma 2.4.2. *Let X be a generic bidegree $(2, 4)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^3$. Then there exists a projective family of bidegree $(2, 4)$ hypersurfaces $\mathfrak{X} \rightarrow B$, containing X as a generic fiber, such that*

- \mathfrak{X} is smooth, and the generic fiber of $\mathfrak{X} \rightarrow B$ is smooth;
- $\forall n_0 \in \mathbb{Z}$, there exists a point $b_{n_0} \in B$ such that the fiber $X_{n_0} := \mathfrak{X}_{b_{n_0}}$ is singular with only finitely many nodes and satisfies
 - (a) the specialization $\ell_{n_0} \subset X_{n_0}$ passes through exactly one node while other specializations $\ell_n \subset X_{n_0}$ do not pass through any nodes for $n \neq n_0$;

(b) all $\ell_n \subset X_{n_0}$ are infinitesimally rigid.

(The notation \mathfrak{X} is different from \mathcal{X} in §2.)

Proof. Consider X as a deformation of the X_0 constructed in Lemma ??, where ℓ_n does not meet singular locus of X_0 . Let $|\mathfrak{L}'|$ be the linear system of bidegree $(2, 4)$ hypersurfaces containing S . The idea of the proof comes from an observation that the space of bidegree $(2, 4)$ hypersurfaces satisfying condition (a) is a divisor in $|\mathfrak{L}'|$.

Indeed, we can give an explicit construction as in [?]. With the notation in Lemma ??, we first consider the one parameter family of bidegree $(2, 4)$ hypersurfaces defined by the equation

$$l_u(x)q(x) + x_3f(x) = 0, \quad (2.10)$$

where $l_u(x) = u_0l_0(x) + u_1l_1(x)$, $u \in \mathbb{P}^1$ defines a linear pencil of bidegree $(1, 1)$ hypersurfaces.

Let C be the curve defined by

$$q(x) = f(x) = 0, \quad (2.11)$$

meeting ℓ_n transversally at distinct points for a generic choice of $f(x)$. Then one can choose $l_u(x)$ outside a countable union of hypersurfaces in the space of all pencils, such that the hyperplane $l_u(x) = 0$ meets S transversely and does not contain more than one point of the countable set

$$C \cap \left(\bigcup_n \ell_n \right).$$

Then the two parameter family

$$\mathfrak{X} = \{l_u(x)q(x) + x_3f(x) + \lambda F(x) = 0\} \rightarrow \mathbb{P}^1 \times \Delta \quad (2.12)$$

will be the desired degeneration for generic $F(x)$. According to our construction, for each integer n_0 , one can find a point $u_{n_0} \in \mathbb{P}^1$ such that $\mathfrak{X}_{(u_{n_0}, 0)}$ satisfies condition (a).

To complete the proof, it remains to show that all ℓ_n are infinitesimal rigid in X_{n_0} . When $n \neq n_0$, the rigidity of ℓ_n comes from the same argument as in the proof of Lemma ??.

If $n = n_0$, let X'_{n_0} be the blow up of X_{n_0} along P , and X''_{n_0} the blow up of X_{n_0} along S . Note that X'_{n_0} and X''_{n_0} are the two small resolutions of X_{n_0} . It suffices to show that the strict transforms ℓ'_{n_0} and ℓ''_{n_0} of ℓ_{n_0} in X'_{n_0} and X''_{n_0} , respectively, are infinitesimally rigid.

Note that ℓ'_{n_0} is still contained in $S \subset X'_{n_0}$ as an exceptional curve, so one can conclude that

$$\mathcal{N}_{\ell'_{n_0}/X'_{n_0}} = \mathcal{O}_{\ell'_{n_0}}(-1) \oplus \mathcal{O}_{\ell'_{n_0}}(-1) \quad (2.13)$$

from the exact sequence (??).

Next, if one can find a special case of X_{n_0} such that

$$\mathcal{N}_{\ell''_{n_0}/X''_{n_0}} = \mathcal{O}_{\ell''_{n_0}}(-1) \oplus \mathcal{O}_{\ell''_{n_0}}(-1), \quad (2.14)$$

then semicontinuity will ensure that (??) holds for the generic case. The existence of such X_{n_0} is known by Lemma 9 in [?], which completes the proof.

2.4.3 Deforming the section through a node

With the notation from the previous section, let $\pi : \mathcal{X} \rightarrow \Delta$ be the restriction $\mathfrak{X}|_{\{u_{n_0}\} \times \Delta}$, whose central fiber is $\pi^{-1}(0) = \mathcal{X}_0 = X_{n_0}$.

If $m \neq n_0$, we know that the section $\ell_m \subset X_{n_0}$ deforms to a section $\ell_m(s)$ of \mathcal{X}_s

and hence yields a codimension two cycle $\mathcal{L}_m \subset \mathcal{X}$ with

$$\mathcal{L}_m \cdot \mathcal{X}_s = \ell_m(s), \quad s \in \Delta. \quad (2.15)$$

However, ℓ_{n_0} in X_{n_0} cannot deform with X_{n_0} in \mathcal{X} since there is a non-trivial obstruction for first order deformations. This obstruction will vanish after a degree two base change. In this subsection, we will show that this is a sufficient condition to deform ℓ_{n_0} with X_{n_0} . The following result is inspired by [?].

Theorem 2.4.4. *The section $\ell_{n_0} \subset X_{n_0}$ can deform with X_{n_0} in \mathcal{X} only after a degree two base change. In other words, we have the following diagram*

$$\begin{array}{ccccc} \mathcal{L}_{n_0} & \xhookrightarrow{i} & \tilde{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \tilde{\pi} & & \downarrow \pi \\ \tilde{\Delta} & \xrightarrow{Id} & \tilde{\Delta} & \xrightarrow{d} & \Delta \end{array} \quad (2.16)$$

where $d: \tilde{\Delta} \rightarrow \Delta$ is the double covering map of the disc Δ ramified at the center $0 \in \Delta$, and $\mathcal{L}_{n_0} \cap \tilde{\pi}^{-1}(0) = \ell_{n_0}$.

Before proceeding to the proof, let us fix some notation as follows:

- X_{n_0} is defined by the equation $F_0(x) = 0$, without loss of generality, having a node $p_0 = (1, 0; 1, 0, 0, 0) \in \mathbb{P}^1 \times \mathbb{P}^3$;
- the section $\ell_{n_0} \subset X_{n_0}$ passing through p_0 is parametrized by the morphism

$$\begin{aligned} \phi: \mathbb{P}^1 &\longrightarrow X_{n_0} \\ t = (t_0, t_1) &\mapsto (t_0, t_1; \phi_0(t), \dots, \phi_3(t)) \end{aligned} \quad (2.17)$$

with $\phi(1, 0) = p_0$ for some degree d homogenous polynomials $\phi_i(t)$,

$i = 0, \dots, 3$;

- the family $\mathcal{X} \rightarrow \Delta$ is given by the equation

$$F_0(x) + sF(x) = 0, \quad s \in \Delta, \quad (2.18)$$

for some polynomial $F(x)$, with $F(p_0) \neq 0$.

With the notation above, we first give an explicit description of the global sections of the normal sheaf $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$.

Lemma 2.4.5. *A global section of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ can be represented by a set of homogenous polynomials*

$$\{(\sigma_i(t))_{i=0,1}, (\delta_j(t))_{j=0,\dots,3}\} \quad (2.19)$$

with $\deg(\sigma_i) = 1$ and $\deg(\delta_j) = d$, subject to the condition

$$\sum_{i=0}^1 \sigma_i(t) \frac{\partial F_0}{\partial t_i}(\phi(t)) + \sum_{j=0}^3 \delta_j(t) \frac{\partial F_0}{\partial x_j}(\phi(t)) = 0. \quad (2.20)$$

Moreover, $(??)$ is a trivial section of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ if and only if it satisfies the condition

$$\delta_j(t) = \sigma_0(t) \frac{\partial \phi_j}{\partial t_0} + \sigma_1(t) \frac{\partial \phi_j}{\partial t_1}, \quad j = 0, \dots, 3. \quad (2.21)$$

Proof. Let us denote the invertible sheaf $\pi_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^3}(b)$ by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(a, b)$, where π_1 and π_2 are natural projections of $\mathbb{P}^1 \times \mathbb{P}^3$. Let \mathcal{T}_X be the tangent sheaf of X . Due to the exact sequence

$$0 \rightarrow \mathcal{O}_{X_{n_0}}^{\oplus 2} \rightarrow \mathcal{O}_{X_{n_0}}^{\oplus 2}(1, 0) \oplus \mathcal{O}_{X_{n_0}}^{\oplus 4}(0, 1) \rightarrow \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^3}|_{X_{n_0}} \rightarrow 0 \quad (2.22)$$

and

$$0 \rightarrow \mathcal{T}_{X_{n_0}} \rightarrow \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^3}|_{X_{n_0}} \rightarrow \mathcal{O}_{X_{n_0}}(2, 4) \rightarrow 0 \quad (2.23)$$

one can express a global section of $\mathcal{T}_{X_{n_0}}$ as a set of bidegree homogenous polynomials $\{(\sigma_i)_{i=0,1}; (\delta_j)_{j=0,\dots,3}\}$ satisfying

$$\sum_{i=0}^1 \sigma_i \frac{\partial F_0}{\partial t_i} + \sum_{j=0}^3 \delta_j \frac{\partial F_0}{\partial x_j} = 0, \quad (2.24)$$

where σ_i are of bidegree $(1, 0)$, while δ_j are of bidegree $(0, 1)$.

Then the statement follows from the following exact sequence,

$$\mathcal{T}_{\ell_{n_0}} \rightarrow \mathcal{T}_{X_{n_0}|_{\ell_{n_0}}} \rightarrow \mathcal{N}_{\ell_{n_0}/X_{n_0}} \longrightarrow 0 \quad (2.25)$$

where the induced map $g : H^0(\ell_{n_0}, \mathcal{T}_{\ell_{n_0}}) \rightarrow H^0(\ell_{n_0}, \mathcal{T}_{X_{n_0}|_{\ell_{n_0}}})$ can be expressed as

$$a_0 \frac{\partial}{\partial t_0} + a_1 \frac{\partial}{\partial t_1} \longmapsto (a_0, a_1; a_0 \frac{\partial \phi_i}{\partial t_0} + a_1 \frac{\partial \phi_i}{\partial t_1})_{i=0, \dots, 3} \quad (2.26)$$

Proof of Theorem ??. Let us make the base change $\tilde{\Delta} \rightarrow \Delta$ sending r to r^2 , and write

$$\tilde{\mathcal{X}} := \{F_0(x) + r^2 F(x) = 0, \quad r \in \tilde{\Delta}\}. \quad (2.27)$$

To prove the assertion, it suffices to show the existence of a formal deformation $\Phi(r, t)$ in (??), i.e. there is a sequence of maps

$$\phi^{[k]}(t) = (t; \phi_0^{[k]}(t), \dots, \phi_3^{[k]}(t)) \in \mathbb{P}^1 \times \mathbb{P}^3, \quad k \geq 0, \quad (2.28)$$

with $\deg(\phi_i^{[k]}(t)) = d$ and $\phi^{[0]} = \phi$, such that the power series

$$\Phi(r, t) = (t_0, t_1; \sum_{k=0}^{\infty} r^k \phi_i^{[k]}(t))_{i=0, 1, \dots, 3} \quad (2.29)$$

satisfies the condition

$$F_0(\Phi(r, t)) + r^2 F(\Phi(r, t)) = 0. \quad (2.30)$$

Our proof of the existence of $\Phi(r, t)$ proceeds as follows:

(I) *First order deformation.* The first order deformation of ϕ is determined by $\phi^{[1]}(t)$, which can be solved by differentiating (??) with respect to r and setting $r = 0$. Hence we obtain

$$\sum_{i=0}^3 \frac{\partial F_0}{\partial x_i}(\phi(t)) \phi_i^{[1]}(t) = 0. \quad (2.31)$$

Note that (??) is a homogenous polynomial of degree $4d+2$, which has $4d+3$ coefficients and the coefficient of the t_0^{4d+2} term is zero by assumption. Then one can consider (??) as $(4d+2)$ equations in $4(d+1)$ unknowns and denote $M_{(\phi, F_0)}$ by the $(4d+2) \times (4d+4)$ matrix corresponding to the system of these equations.

Our first claim is that the $M_{(\phi, F_0)}$ is of full rank, which is equivalent to saying that the dimension of the solution space of $\phi^{[1]}(t)$ is two.

By Lemma ??, the set

$$\{(t_0, t_1); \phi_0^{[1]}, \dots, \phi_i^{[1]}\} \quad (2.32)$$

gives a global section of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ and is trivial if and only if

$$\phi_i^{[1]} = t_0 \frac{\partial \phi_i}{\partial t_0} + t_1 \frac{\partial \phi_i}{\partial t_1} \quad (2.33)$$

by (??). So if $\text{rank } M_{(\phi, F_0)} \leq 4d+1$, then $\dim H^0(\ell_{n_0}, \mathcal{N}_{\ell_{n_0}/X_{n_0}}) \geq 2$.

Let $ev : H^0(\ell_{n_0}, \mathcal{N}_{\ell_{n_0}/X_{n_0}}) \rightarrow \mathbb{C}^3$ be the evaluation map at p_0 . As in [?] §3, one can show that there is at most one condition lifting the analytic section of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ to a section of $\mathcal{N}_{\ell'_{n_0}/X'_{n_0}}$, because the image of ev is at most two dimensional, while the composition of the sequence of evaluation maps at p_0

$$\mathcal{N}_{\ell'_{n_0}/X'_{n_0}} \rightarrow \mathcal{N}_{\ell_{n_0}/X_{n_0}} \rightarrow \mathbb{C}^3 \quad (2.34)$$

only has a one dimensional image.

However, from (??), we know that $H^0(\ell'_{n_0}, \mathcal{N}_{\ell'_{n_0}/X'_{n_0}}) = 0$. Thus we prove the first claim by contradiction.

(II) *Higher order.* We continue to solve $\phi^{[2]}(t)$ by differentiating (??) twice, and thus obtain

$$\sum_{i=0}^3 \frac{\partial F_0}{\partial x_i} (\phi(t)) \phi_i^{[2]}(t) = - \sum_{i,j} \frac{\partial F_0}{\partial x_i \partial x_j} (\phi(t)) \phi_i^{[1]}(t) \phi_j^{[1]}(t) + 2F(\phi(t)). \quad (2.35)$$

Obviously, there is a non-trivial obstruction to lift $\phi^{[1]}(t)$ to second order given by the equation,

$$\sum_{i,j} \frac{\partial F_0}{\partial x_i \partial x_j}(p_0) \phi_i^{[1]}(1,0) \phi_j^{[1]}(1,0) = -2F(p_0) \neq 0. \quad (2.36)$$

Any $\phi^{[1]}(t)$ satisfying (??) can be lifted to the second order, because M_{ϕ, F_0} is full rank.

Our second claim is that there exists a first order deformation which can be lifted to second order. Otherwise, every $\phi^{[1]}(t)$ will satisfy the condition

$$\sum_{i,j} \frac{\partial F_0}{\partial x_i \partial x_j}(p_0) \phi_i^{[1]}(1,0) \phi_j^{[1]}(1,0) = 0. \quad (2.37)$$

From the above assumption, there is a non-trivial analytic section $\alpha \in H^0(\ell_{n_0}, \mathcal{N}_{\ell_{n_0}/X_{n_0}})$, whose image via the evaluation map at p_0 lies in the tangent cone C_{p_0} of X_{n_0} at p_0 and is normal to the tangent direction of ℓ_{n_0} .

As in [?] §3, this means that $ev(\alpha)$ lies in the union of the images of $\mathcal{N}_{\ell'_{n_0}/X'_{n_0}}$ and $\mathcal{N}_{\ell''_{n_0}/X''_{n_0}}$ in $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$. This is a contradiction, because none of the non-trivial sections of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ can lift by Lemma ??. Hence there exists $\bar{\phi}^{[1]}(t)$ satisfying (??), and the second claim is proved.

Furthermore, set $b_i = \bar{\phi}_i^{[1]}(1,0)$. It is not difficult to see that the equation (??) along with

$$\sum_{i,j} \frac{\partial F_0}{\partial x_i \partial x_j}(p_0) \phi_i^{[1]}(1,0) b_j = 0 \quad (2.38)$$

only has a one dimensional set of solutions. Hence the associated $(4d+3) \times (4d+4)$ matrix $M'_{\bar{\phi}^{[1]}, F_0}$ is full rank.

For higher orders, $\phi^{[n]}(t)$ is determined by the equation

$$\sum_{i=0}^3 \frac{\partial F_0}{\partial x_i}(\phi(t)) \phi_i^{[n]}(t) = \text{some polynomial given by } \phi^{[k]} \text{ for } k < n, \quad (2.39)$$

while the obstruction to $(n+1)$ th order is

$$\sum_{i,j} \frac{\partial F_0}{\partial x_i \partial x_j}(p_0) \phi_i^{[n]}(1,0) b_j = \text{some number given by } \phi^{[k]} \text{ for } k < n. \quad (2.40)$$

Then one can solve $\phi^{[n]}(t)$ by induction because of the full rank of $M'_{\phi^{[1]}, F_0}$.

2.4.6 Semistable degeneration.

In this subsection, we will desingularize the family $\tilde{\mathcal{X}}$ to obtain a semistable degeneration, and identify the group of components associated to this semistable degeneration.

Let \mathcal{W} be the blow up of $\tilde{\mathcal{X}}$ along all the nodes on $\tilde{\mathcal{X}}_0$. Then we have

- the ambient space \mathcal{W} is smooth, and the generic fiber of $\mathcal{W} \rightarrow \tilde{\Delta}$ is smooth;
- the central fiber $\mathcal{W}_0 = \bigcup_{i=0}^{18} W_i$ is strictly normal crossing, where
 - (1) W_0 is the blow up of $\tilde{\mathcal{X}}_0$ along all the nodes;
 - (2) W_i are disjoint smooth quadratic threefolds in \mathbb{P}^4 , meeting W_0 transversally at the exceptional divisor $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, \dots, 18$.

As an application of Theorem ??, we shall apply (??) to compute the group of components of the Néron model $\bar{J}(\mathcal{W})$ associated to the semistable degeneration $\mathcal{W} \rightarrow \tilde{\Delta}$.

In order to give a geometric description of the homology groups of each component of \mathcal{W}_0 , we first set the following notation:

- P is the strict transform of bidegree $(1,0)$ hyperplane section of X_{n_0} in W_0 , and D is a generic fiber of W_0 over \mathbb{P}^1 ; \tilde{H} is the strict transform of H in W_0 ;
- E_i are exceptional divisors of W_0 , $i = 1, 2, \dots, 18$.
- Q_i is the hyperplane section of W_i , $i = 1, 2, \dots, 18$.

- L is a line on the fiber of W_0 over \mathbb{P}^1 , and L' is a section of W_0 with respect to the projection; C' is the proper transform of the curve (??) in W_0 ; R_i is one of the ruling of E_i .
- L_i is the line in W_i .

Then the integral basis for these homology groups can be represented by the fundamental class of the algebraic cycles above:

$$\begin{aligned} H_2(W_0) &= \langle [L], [L'], [C'], [R_1], \dots, [R_{18}] \rangle, & H_4(W_0) &= \langle [D], [K], [\tilde{P}], [E_1], \dots, [E_{18}] \rangle; \\ H_2(W_i) &= \langle [L_i] \rangle, \quad i = 1, 2, \dots, 18; & H_4(W_i) &= \langle [Q_i] \rangle, \quad i = 1, 2, \dots, 18. \end{aligned} \quad (2.41)$$

By a straightforward computation, we can express the map

$$\mu_{\mathbb{Z}} : \bigoplus_{i=0}^{18} H_4(W_i, \mathbb{Z}) \longrightarrow \bigoplus_{i=0}^{18} H_2(W_i, \mathbb{Z})$$

as a matrix:

	$[P]$	$[D]$	$[\tilde{H}]$	$[E_i]$	$[Q_j]$
$[L]$	1	0	1	0	0
$[L']$	0	1	1	0	0
$[C']$	0	0	18	1	0
$[R_k]$	0	0	1	$2\delta_{ik}$	$2\delta_{jk}$
$[L_l]$	0	0	1	$2\delta_{il}$	$2\delta_{jl}$

Thus the group is computed as

$$G = \frac{\text{Im}(\mu_{\mathbb{Q}})_{\mathbb{Z}}}{\text{Im}(\mu_{\mathbb{Z}})} = \frac{\bigoplus_{k=1}^{18} \mathbb{Z}([R_k] + [L_k])}{\bigoplus_{k=1}^{18} \mathbb{Z}(2[R_k] + 2[L_k]) \oplus \mathbb{Z} \sum_{i=1}^{18} ([R_i] + [L_i])}. \quad (2.42)$$

Furthermore, let $\check{\mathcal{L}}_{n_0}$ denote the strict transform of \mathcal{L}_{n_0} in \mathcal{W} . The following lemma is straightforward (cf. [?]) and will be used in the next section.

Lemma 2.4.7. *Let E_{i_0} be the exceptional divisor in W_0 corresponding to the node which ℓ_{n_0} passes through, and $\check{\ell}_{n_0}$ the strict transform of ℓ_{n_0} in W_0 . Then*

$$\check{\mathcal{L}}_{n_0} \cap W_0 = \check{\ell}_{n_0} + (\text{one of the rulings of } E_{i_0}). \quad (2.43)$$

2.5 Infinite generation of the subgroup of Griffiths group

2.5.1 Griffiths group of Calabi-Yau threefolds

Let X be a smooth projective threefold and $\text{CH}^2(X)_{\text{alg}}$ be the subgroup of $\text{CH}^2(X)$ consisting of codimension 2 cycles which are algebraically equivalent to zero. The Abel-Jacobi image

$$\text{AJ}_X(\text{CH}^2(X)_{\text{alg}}) = A \subseteq J(X) \quad (2.44)$$

is an abelian variety. The abelian variety A is called the algebraic part of $J(X)$. It lies in the largest complex subtorus $J(X)_{\text{alg}} \subset J(X)$, whose tangent space at 0 is contained in $H^{1,2}(X)$. (cf. [?].vI)

The Griffiths group $\text{Griff}^2(X)$ is the quotient $\text{CH}^2(X)_{\text{hom}}/\text{CH}^2(X)_{\text{alg}}$, which is a subgroup of $\mathcal{A}^2(X)$. In the case of Calabi-Yau threefolds, the following result is mentioned in [?] and [?].

Theorem 2.5.2. *If X is a nonrigid Calabi-Yau threefold, i.e. $h^1(\mathcal{T}_X) \neq 0$, then $J(X_s)_{\text{alg}} = 0$ for a general deformation X_s of X . In particular, AJ_{X_s} factors*

$$\text{AJ}_{X_s} : \text{Griff}^2(X_s) \rightarrow J(X_s)$$

Now we return to the case of X a generic bidegree $(2, 4)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^3$, and hence $\text{AJ}_X : \text{Griff}^2(X) \rightarrow J(X)$ is well defined. Recalling that the group $\mathcal{A} \subset \mathcal{A}^2(X)$ is generated by $\{\ell_n\}$ of different degree, we consider the non-trivial group $\mathcal{A} \cap \text{Griff}^2(X)$.

Since the rank of $H_2(X, \mathbb{Z})$ is two by the Lefschetz hyperplane theorem, there exists integers a, a_n, b_n , such that

$$\psi_n := a\ell_n - a_n\ell_0 - b_n\ell_1 \equiv_{hom} 0, \quad \forall n \in \mathbb{Z}. \quad (2.45)$$

As in Remark 3.3, the fundamental class of ℓ_n is of type $(1, d_n)$ in $H_2(X, \mathbb{Z})$ and we can assume ℓ_0 is of type $(1, 0)$. After a suitable choice of ℓ_1 , we can select a to be odd. In fact, let us denote κ by the largest number such that d_n is divisible by 2^κ for all $n \in \mathbb{Z}$, and denote ℓ_1 by the section of type $(1, d_1)$ satisfying that $d_1/2^\kappa$ is odd, one can choose $a = d_1/2^\kappa$ as desired. Denote \mathcal{G} by the subgroup of $\mathcal{A} \cap \mathcal{A}^2(X)$ generated by ψ_n .

2.5.3 Infinite generation of \mathcal{G}

In this subsection, we shall prove the following result, which implies Theorem ??.

Theorem 2.5.4. *The Abel-Jacobi image $\text{AJ}_X(\mathcal{G}) \otimes \mathbb{Q}$ is of infinite rank for generic X .*

Proof. Suppose that there is a relation

$$\sum_{finite} c_n \text{AJ}_X(\psi_n) = 0 \quad (2.46)$$

for generic X , which gives $\text{AJ}_X(\sum_{finite} ac_n \ell_n) = 0$. In particular, we assume that (??) holds for generic fiber of the two parameter family $\mathfrak{X} \rightarrow \mathbb{P}^1 \times \Delta$ (??).

From the construction in Lemma ??, for each integer n , there is a point $(u_n, 0) \in \mathbb{P}^1 \times \Delta$ such that the fiber $X_n = \mathfrak{X}_{(u_n, 0)}$ over $(u_n, 0)$ satisfies the conditions (a) and (b). Let $\mathcal{X} \rightarrow \Delta$ be the restriction $\mathfrak{X}|_{\{u_n\} \times \Delta}$ and $\tilde{\mathcal{X}} = \mathcal{X} \times_\Delta \tilde{\Delta}$ for a degree two base change $\tilde{\Delta} \xrightarrow{r \mapsto r^2} \Delta$. Then the family of cycles

$$\mathcal{Z} = ac_n \mathcal{L}_n + \sum_{m \neq n} ac_m \tilde{\mathcal{L}}_m \quad (2.47)$$

satisfies $\text{AJ}_{\tilde{\mathcal{X}}_r}(\mathcal{Z} \cdot \tilde{\mathcal{X}}_r) = 0$, where \mathcal{L}_n is given by (??) in Theorem ?? and $\tilde{\mathcal{L}}_m$ is the lift of ?? in $\tilde{\mathcal{X}}$ for $m \neq n$.

To make use of the Néron model, we blow up $\tilde{\mathcal{X}}$ along all the nodes to get the semistable degeneration $\mathcal{W} \rightarrow \tilde{\Delta}$ as in § 4.6. There is an associated Néron model $\bar{J}(\mathcal{W})$. Denote by $\check{\mathcal{Z}}$ the lifting of \mathcal{Z} in \mathcal{W} ; then the associated admissible normal function $\nu_{\check{\mathcal{Z}}}$ is a zero holomorphic section and naturally extends to the identity component of $\bar{J}(\mathcal{W})$.

On the other hand, write $\check{Z}_0 := \check{\mathcal{Z}} \cdot \mathcal{W}_0$; then $\nu_{\check{\mathcal{Z}}}$ extends to the component corresponding to the class of $[\check{Z}_0]$ in $\bigoplus_{i=0}^{18} H_2(W_i, \mathbb{Z})$ by Theorem ??. According to Lemma ??, we have

$$[\check{Z}_0] = ac_n([R_n] + [L_n]) + \text{linear combinations of } [L], [L'] \text{ and } [C'] \quad (2.48)$$

which corresponds to $ac_n([R_n] + [L_n])$ in G . Then as indicated in §4.6, $\nu_{\check{\mathcal{Z}}}$ extends to the identity component if and only if c_n is even, because a is odd.

Repeating the above process for each integer n , one proves that all the coefficients in (??) are even. Thus, the elements $\{\text{AJ}_X(\psi_n)\}$ are linearly independent modulo two which implies that $\text{AJ}_X(\mathcal{G}) \otimes \mathbb{Z}_2$ has infinite rank. Then the assertion follows from the lemma below.

Lemma 2.5.5. *Let M be an abelian group with*

$$M_{\text{torsion}} \subseteq (\mathbb{Q}/\mathbb{Z})^r.$$

Then $\text{rank}_{\mathbb{Z}_2}(M \otimes \mathbb{Z}_2) \leq \text{rank}_{\mathbb{Q}}(M \otimes \mathbb{Q}) + r$.

Remark 2.5.6. *Note that the monodromy of the degeneration $\mathcal{W} \rightarrow \tilde{\Delta}$ satisfies $(T - I)^2 = 0$. One can take an alternate approach by using Clemens' Néron model $\bar{J}^{\text{Clemens}}(\mathcal{W})$ [?] to extend the associated normal function. Actually, the identity component of $\bar{J}(\mathcal{W})$ is a subspace of Clemens's Néron model (cf. see also [?]).*

Remark 2.5.7. Let $\iota : \mathcal{W} \rightarrow \mathcal{W}$ be the natural involution. The admissible normal function $\nu_{\mathcal{Z}'}$ associated to the family of algebraic cycles $\mathcal{Z}' = \check{\mathcal{L}}_n - \iota(\check{\mathcal{L}}_n)$ extends to the identity component [?]. One can also prove the infinite generation of G by showing that $\nu_{\mathcal{Z}'}(0)$ is a nontrivial element in the identity component. The proof is similar to our computation of the group of components (cf. [?]).

2.6 Proof of the Main theorem

Proof of Theorem ??. Assume to the contrary that the union of the sections $\{\ell_n\}$ is not Zariski dense in X . Let Σ be the Zariski closure of the union of these curves, and $\tilde{\Sigma}$ the desingularization of Σ . Then the proper morphism

$$\varphi : \tilde{\Sigma} \rightarrow X$$

induces a homomorphism

$$\varphi_* : \mathcal{A}^1(\tilde{\Sigma}) \rightarrow \mathcal{A}^2(X).$$

The homomorphism φ_* maps the algebraic cycle ℓ_n in $\tilde{\Sigma}$ to the corresponding 1-cycle in X . So the group \mathcal{A} is contained in the image of $\mathcal{A}^1(\tilde{\Sigma})$ via φ_* .

It is well known that $\mathcal{A}^1(\tilde{\Sigma}) = NS((\tilde{\Sigma}))$ is a finitely generated abelian group by the Néron-Severi theorem, which contradicts Theorem ??. This completes the proof.

Remark 2.6.1. Our result can be generalized to other Calabi-Yau threefolds fibered by complete intersection K3 surfaces in \mathbb{P}^n . For instance, one can find an analogous statement for some Calabi-Yau threefolds in $\mathbb{P}^1 \times \mathbb{P}^4$ fibered by the complete intersection of a quadratic and a cubic in \mathbb{P}^4 .

Remark 2.6.2. Our method does not yield examples in $\bar{\mathbb{Q}}(t)$, since all type (2,4) hypersurfaces X over $\bar{\mathbb{Q}}$ might lie in a countable union of “bad” hypersurfaces of the parameterization space.

Furthermore, one can also see [?] [?] for conjectures of $CH_0(Y)_{hom}$ when Y is a surface over a number field or a function field of a curve defined over a finite field.

2.7 Higher dimensional Calabi-Yau hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^N$

In this section, we consider the case of bidegree $(2, N+1)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^N$ for $N \geq 3$. The following theorem is obtained via a similar argument as Lemma ??.

Theorem 2.7.1. *For a general hypersurface $X^N \subset \mathbb{P}^1 \times \mathbb{P}^N$ of bidegree $(2, N+1)$, there exists an infinite sequence of sections $\{\ell_k\}$ on X^N of different degrees with respect to the projection $X^N \rightarrow \mathbb{P}^1$, such that*

$$\mathcal{N}_{\ell_k/X^N} = \mathcal{O}_{\ell_k}(-1) \oplus \mathcal{O}_{\ell_k}(-1) \oplus \overbrace{\mathcal{O}_{\ell_k} \oplus \dots \oplus \mathcal{O}_{\ell_k}}^{N-3}. \quad (2.49)$$

Furthermore, the subgroup $\mathcal{G}^N \subset \mathcal{A}^2(X^N)$ generated by the algebraic codimension 2-cycles ψ_k^N , which are swept out by the deformations of ℓ_k , is not finitely generated.

Proof. The proof will proceed by induction on N . Suppose our statement holds for $N = m \geq 3$. When $N = m+1$, it suffices to produce a nodal hypersurface of bidegree $(2, m+2)$ with the desired properties. The construction is as follows,

Let us denote the coordinate of $\mathbb{P}^1 \times \mathbb{P}^{m+1}$ by $x = (s, t; x_0, \dots, x_{m+1})$. Then we consider the bidegree $(2, m+2)$ hypersurface X_0^{m+1} defined by the equation

$$x_0 g(x) + x_n h(x) = 0 \quad (2.50)$$

for some bidegree $(2, m+1)$ polynomials g, h .

Now, we choose our $g(x), h(x)$ satisfying the following conditions:

(1) X_0^{m+1} is only singular at

$$x_0 = x_{m+1} = g(x) = h(x) = 0; \quad (2.51)$$

(2) the subvariety

$$X^m := \{x_{m+1} = g(x) = 0\} \quad (2.52)$$

satisfies the inductive assumption. Denote by ℓ_k the corresponding sections on X^m ;

(3) All the sections ℓ_k lie outside the singular locus (??).

Similar as in the proof of Lemma ??, condition (1) will be satisfied due to Bertini's theorem and condition (3) can be achieved for a generic choice of $g(x)$ outside countably many hypersurfaces of the parametrization space of bidegree $(2, m)$ polynomials.

Next, we compute the normal bundle $\mathcal{N}_{\ell_k/X_0^{m+1}}$ from the following exact sequence:

$$0 \rightarrow \mathcal{N}_{\ell_k/X^m} \rightarrow \mathcal{N}_{\ell_k/X_0^{m+1}} \rightarrow \mathcal{N}_{X^m/X_0^{m+1}}|_{\ell_k} \rightarrow 0. \quad (2.53)$$

By assumption, we have $\mathcal{N}_{\ell_k/X^m} = \mathcal{O}_{\ell_k}(-1)^{\oplus 2} \oplus \mathcal{O}_{\ell_k}^{\oplus m-3}$. Since

$$\mathcal{N}_{X^m/X_0^{m+1}}|_{\ell_k} = \mathcal{O}_{\ell_k},$$

it follows that

$$\mathcal{N}_{\ell_k/X_0^{m+1}} = \mathcal{O}_{\ell_k}(-1)^{\oplus 2} \oplus \mathcal{O}_{\ell_k}^{\oplus m-2}. \quad (2.54)$$

Moreover, let $\mathcal{G}^{m+1} \subset \mathcal{A}^2(X_0^{m+1})$ be the subgroup generated by deformations of ℓ_k in X_0^{m+1} . There is a morphism

$$i^* : \mathcal{A}^2(X_0^{m+1}) \rightarrow \mathcal{A}^2(X^m). \quad (2.55)$$

induced by the inclusion $X^m \xrightarrow{i} X_0^{m+1}$. Then the infinite generation of \mathcal{G}^{m+1} follows from the inductive assumption on X^m . This completes the proof.

As in §2.6, the following result is deduced from the infinite generation of \mathcal{G}^N :

Corollary 1. *The sections on general bidegree $(2, N+1)$ hypersurfaces of $\mathbb{P}^1 \times \mathbb{P}^N$ are Zariski dense.*

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